ELSEVIER

Linear Algebra and its Applications 348 (2002) 7–21

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree

Charles R. Johnson [a,1], António Leal Duarte [b,*,2]

[a] *Department of Mathematics, College of William and Mary, Williamsburg, VA 23185, USA*
[b] *Dep. de Matemática, University de Coimbra, Apartado 3008, 3000 Coimbra, Portugal*

Received 6 July 2000; accepted 3 October 2001

Submitted by J. Dias da Silva

Abstract

We consider the general problem of determining which lists of multiplicities for the eigenvalues occur among Hermitian matrices the graph of whose off-diagonal entries is a given tree. Several restrictions are cited and a construction strategy is given. Together, these are sufficient to characterize all lists for each tree in two infinite classes: the *double paths* and *generalized stars*, and to tabulate all lists for trees on fewer than nine vertices. Such tables should be useful for formulating and dispelling general conjectures. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Hermitian matrices; Eigenvalues; Multiplicities; Trees; Parter vertices

1. Introduction

The (undirected) graph $G = G(A)$, on vertices $\{1, 2, \ldots, n\}$, of an n-by-n Hermitian matrix $A = (a_{ij})$ has an edge $\{i, j\}$ if and only if $a_{ij} \neq 0$. The diagonal entries of A, which may or may not be 0, are not taken into account. We consider the set of all Hermitian matrices that share a common graph G:

$$H(G) = \{A : A = A^*, G(A) = G\}.$$

The general question, in which we are interested, is which lists of multiplicities occur for the distinct eigenvalues of A, as A runs over $H(G)$, for a given undirected

* Corresponding author.

[1] This research was supported by Project 574/94 of Fundação Luso Americana para o Desenvolvimento.

[2] This research was supported by CMUC and by Projecto PRAXIS/2/2.1/Mat/458/94.

PII: S0024-3795(01)00522-5

graph G. (There are natural analogous problems for directed graphs, real or complex matrices and either algebraic or geometric multiplicity, which we do not address here.)

Specifically if A is an n-by-n Hermitian matrix with distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, we associate the *ordered* partition

$$p(A) = (p_1, p_2, \ldots, p_k)$$

with A, in which $p_i = m_A(\lambda_i)$, $i = 1, \ldots, k$. Here, as throughout, $m_A(\lambda)$ denotes the multiplicity of the eigenvalue λ in the Hermitian matrix A. Thus $\sum_{i=1}^{k} p_i = n$, so p is an (ordered) partition of n into k parts.

For a given (undirected) graph G, we denote the set of all partitions $p(A)$, as A runs through $H(G)$ by $L_0(G)$. We shall actually concentrate on the partitions that occur in $H(G)$ *without* respect to order. If $\tilde{p} = \tilde{p}(A) = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_k)$ is the reordering of $p(A)$ in a natural nonincreasing order, $\tilde{p}_1 \geqslant \tilde{p}_2 \geqslant \cdots \geqslant \tilde{p}_k$, then we denote by $L(G)$ the collection of $\tilde{p}$'s that occur as A runs through $H(G)$.

We assume that G is connected throughout. A very natural place to start studying $L(G)$, then, is the case in which G is minimally connected, i.e. G is a tree T, and we focus our attention on determining $L(T)$ when T is a tree. In the following section we review several results relevant to this question, then prove a new lemma (which generalizes, somewhat, the known facts about tridiagonal matrices) that we need for certain constructions, and then, as we are far from a general solution, determine $L(T)$ for two large classes of trees: the *generalized stars* and the *double paths*. We use these results to tabulate $L(T)$ for trees on eight vertices.

We note that, upon specializing in trees, our question encompasses rather more general matrices than just Hermitians. For example, the answer is the same if the underlying class is the real symmetrics, the sign symmetrics or the complex matrices such that $a_{ij}\overline{a}_{ji} > 0$ when $\{i, j\}$ is an edge of T.

2. Prior results

It is an important classical result that, when T is a path on n vertices, $L(T)$ contains only the partition $(1, 1, \ldots, 1)$, which, it is an easy exercise to show, is contained, properly, in every other $L(G)$, for G a graph on n vertices. Beginning with [9] several papers have considered important aspects of the general problem of determining $L(T)$ for a tree T [5,6,8–10] (though none previously treated the full problem) and there is a substantial literature on more detailed aspects of spectral theory for paths, i.e. irreducible tridiagonal matrices, e.g. [1] and [3, Section 3] and their references. We mention here three important results from the general literature that will be helpful to us in describing $L(T)$ for certain classes of trees. Clearly, results that constrain the possible partitions in $L(T)$ are needed, and all the results we mention in this section may be viewed in this way. Results that insure the existence of certain partitions in $L(T)$ are also needed and we discuss these later.

A key and powerful fact that may be distilled from, collectively [9,10], though it is not so simply stated there, is the following. We use deg to denote the degree of a vertex in an implicitly understood graph and $A(i)$ to denote the principal submatrix of the matrix A resulting from deleting row and column i.

Theorem 1 [9,10]. *Let A be an n-by-n real symmetric matrix and λ a real number such that* (1) *$G(A)$ is a tree T on $\{1,\ldots,n\}$ and* (2) *$m_A(\lambda) > 1$. Then there is a vertex i of T such that*:
(α) $\deg(i) > 2$,
(β) *λ is an eigenvalue of at least three irreducible components of $A(i)$ and*
(γ) $m_{A(i)}(\lambda) = m_A(\lambda) + 1$.

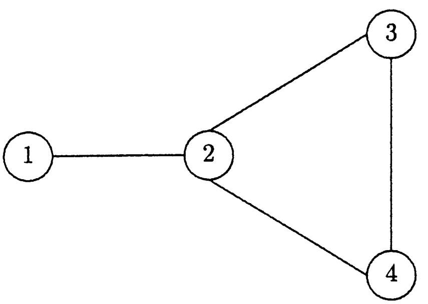
We call such a vertex i, as guaranteed by the theorem, a *Parter vertex* of T for A. The dependence upon A is important, as different matrices with the same graph T could have different Parter vertices for an eigenvalue of the same multiplicity: in fact, several vertices could fulfill the definition of a Parter vertex for the same eigenvalue of the same matrix, and the same vertex could be a Parter vertex for different eigenvalues of the same matrix. If, for example, there is only one vertex of degree more than 2 in T (a "generalized star"), then it must be the Parter vertex for all multiple eigenvalues of A. Such applications of the theorem convey considerable information about the eigenstructure of principal submatrices associated with parts of the tree.

Note that Theorem 1 fails for nontrees.

Example 1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then $G(A)$ is



and $0 \in \sigma(A)$ has multiplicity 2. However $0 \in \sigma(A(i))$ has multiplicity only one for each vertex i (in particular for the only degree 3 vertex, vertex 2).

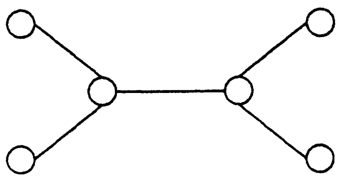
In [5], for T a tree, the maximum multiplicity, $M(T)$, of an eigenvalue of a matrix in $H(T)$ (that is $M(T) = \max_{\tilde{p}\in L(T)} \tilde{p}_1$) was studied. This was related to the path covering number $P(T)$: the fewest vertex disjoint paths of T that cover all vertices of T (a vertex counts as a (degenerate) path). One of the characterizations of $M(T)$ is the following:

Theorem 2 [5]. *For each tree T,*

$$M(T) = P(T),$$

i.e., the maximum multiplicity is the path covering number of T.

The path covering number can be efficiently computed and the theorem constrains $L(T)$ considerably. For example, if T_1 is the following tree,



then $P(T_1) = 2$ and, in fact, $L(T_1) = \{(2, 2, 1, 1), (2, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1)\}$. (See Section 3 on double paths.)

In [6], the minimum number of distinct eigenvalues among matrices in $H(T)$, T a tree, was studied ($q(T) =$ the fewest parts in a partition in $L(T)$) and related to the *diameter* of T, $d(T) =$ the number of vertices in a longest path in T (for trees). The main result was the following inequality:

Theorem 3 [6]. *For each tree T,*

$$q(T) \geqslant d(T),$$

i.e., there are at least the diameter many distinct eigenvalues.

The diameter of a tree is also easily computed and, although we suspect that the above inequality is actually an equality, it also severely restricts $L(T)$. For example, for T_1 above, $d(T_1) = 4$, so that (2, 2, 2) is ruled out of $L(T_1)$, the only partition not ruled out by $P(T_1) = 2$.

Another simple observation should be made here. Since for any tree T, any $A \in H(T)$ is both a translate of a diagonal similarity of an irreducible nonnegative matrix and a translate of an irreducible M-matrix, the largest and the smallest eigenvalues of A each have multiplicity 1. This may be phrased as:

Remark 4. For each tree T, on at least two vertices and for each $p \in L_0(T)$, $p = (p_1, \ldots, p_k)$ we have $p_1 = p_k = 1$.

Alternatively, each $\tilde{p} \in L(T)$ ends with (at least) two 1's. It may happen that all $\tilde{p} \in L(T)$ end with *more* than two 1's. It is not yet known what the minimum number of 1's among $\tilde{p} \in L(T)$ is in terms of T. However, the remark also excludes (2, 2, 2) from $L(T_1)$, T_1 as above.

3. Construction

As we mentioned before, much is known about the eigenvalues of tridiagonal matrices (T is a path) and their submatrices, etc. and this is usually useful in constructing elements of $L(T)$. However, we need somewhat more than what seems to be known, which we describe here.

The following lemma is well known and can be easily proved by a direct calculation.

Lemma 5. *Let f be a monic polynomial of degree n, $n > 1$, having all its roots real and distinct and let g be a monic polynomial with $\deg g < \deg f$. Then g has $n - 1$ distinct real roots strictly interlacing the roots of f if and only if the coefficients of the partial fraction decomposition of g/f are positive real numbers.*

Lemma 6. *Let $\lambda_1, \ldots, \lambda_n$ be n distinct real numbers, g_1, g_2, g_3 be monic polynomials having (collectively) all their roots real and distinct, with $\deg g_1 + \deg g_2 + \deg g_3 = n - 1$. Denote by $\mu_1, \ldots, \mu_{n-1}$ the roots of $g_1 g_2 g_3$ and suppose that*

$$\lambda_1 > \mu_1 > \lambda_2 > \cdots > \mu_{n-1} > \lambda_n. \tag{1}$$

Then there exists $a \in \mathbb{R}$, $x_1, x_2, x_3 \in \mathbb{R}^+$ and monic polynomials h_1, h_2, h_3, with $\deg h_i = (\deg g_i) - 1$, having all their roots real, the roots of h_i strictly interlacing the roots of g_i (if $\deg g_i > 1$) and such that:

$$\prod_{j=1}^{n}(\lambda - \lambda_j) = (\lambda - a)g_1g_2g_3 - x_1h_1g_2g_3 - x_2h_2g_1g_3 - x_3h_3g_1g_2. \tag{2}$$

Proof. From the partial fractional decomposition of

$$\frac{\prod_{j=1}^{n}(\lambda - \lambda_j)}{g_1g_2g_3},$$

we conclude the existence of (unique) real numbers a, x_1, x_2, x_3 and (unique) monic real polinomials h_1, h_2, h_3, with $\deg h_i < \deg g_i$, verifying (2). Let us prove that h_i has $(\deg g_i) - 1$ distinct roots, and that these roots strictly interlace with the roots of g_i.

If $\deg g_i = 1$, there is nothing to prove, so suppose $\deg g_i > 1$. Consider any pair of sucessive roots of g_i, say μ_r and μ_{r+p}. Denote by R the set of roots of $(g_1g_2g_3)/g_i$. Putting $\lambda = \mu_r$ and $\lambda = \mu_{r+p}$ in (2) we have

$$\prod_{j=1}^{n}(\mu_r - \lambda_j) = -x_i h_i(\mu_r)\prod_{\mu\in R}(\mu_r - \mu). \tag{3}$$

$$\prod_{j=1}^{n}(\mu_{r+p} - \lambda_j) = -x_i h_i(\mu_{r+p})\prod_{\mu\in R}(\mu_{r+p} - \mu). \tag{4}$$

The factors $(\mu_r - \mu)$ and $(\mu_{r+p} - \mu)$ have both the same sign for each $\mu \in R$ except for $\mu = \mu_k$ with $k = r+1, \ldots, r+p-1$ (note that the later belongs to R, since μ_r and μ_{r+p} are the sucessive roots of g_i). So when $p-1$ is even the products

$$\prod_{\mu\in R}(\mu_r - \mu) \quad \text{and} \quad \prod_{\mu\in R}(\mu_{r+p} - \mu)$$

have the same sign. From the inequalities (1) follows that, in the case of $p-1$ being even,

$$\prod_{j=1}^{n}(\mu_r - \lambda_j) \quad \text{and} \quad \prod_{j=1}^{n}(\mu_{r+p} - \lambda_j)$$

have opposite signs.

When $p-1$ is odd,

$$\prod_{\mu\in R}(\mu_r - \mu) \quad \text{and} \quad \prod_{\mu\in R}(\mu_{r+p} - \mu)$$

have opposite signs while

$$\prod_{j=1}^{n}(\mu_r - \lambda_j) \quad \text{and} \quad \prod_{j=1}^{n}(\mu_{r+p} - \lambda_j)$$

have the same sign.

So, using (3) and (4), we see that in both cases $h_i(\mu_r)$ and $h_i(\mu_{r+p})$ must have opposite signs. This means that h_i has an odd number of roots in $]\mu_{r+p}, \mu_r[$ for any pair of sucessive roots μ_r and μ_{r+p} of g_i. But $\deg h_i < \deg g_i$, so that odd number must be 1, that is $\deg h_i = (\deg g_i) - 1$ and the roots of h_i strictly interlace with the roots of g_i.

Now let us prove that $x_i > 0$. Suppose that μ_r is the q-greatest root of g_i. From the interlacing property just proved it follows that the sign of $h_i(\mu_r)$ is $(-1)^{q-1}$. The sign of $\prod_{\mu\in R}(\mu_r - \mu)$ is $(-1)^{r-q-2}$ and the sign of $\prod_{j=1}^{n}(\mu_r - \lambda_j)$ is $(-1)^r$ so (3) implies $x_i > 0$. $\square$

Theorem 7. *Let $\gamma_1, \ldots, \gamma_k, \lambda_{k+1}, \ldots, \lambda_n$ be n distinct real numbers and $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$, $p+q = n-2k-1$, be real numbers such that $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$, $\gamma_1, \ldots, \gamma_k$ strictly interlace with $\lambda_{k+1}, \ldots, \lambda_n$. Then, there is a real symmetric irreducible tridiagonal matrix A such that*:

$$\sigma(A) = \{\gamma_1, \dots, \gamma_k, \lambda_{k+1}, \dots, \lambda_n\},$$
$$\sigma\big(A[1, \dots, p+k]\big) = \{\gamma_1, \dots, \gamma_k, \alpha_1, \dots, \alpha_p\} \quad \textit{and}$$
$$\sigma\big(A[n-(q+k)+1, \dots, n]\big) = \{\gamma_1, \dots, \gamma_k, \beta_1, \dots, \beta_q\}.$$

Proof. Let

$$f(\lambda) = \prod_{j=k+1}^{n} (\lambda - \lambda_j),$$

$$g_1(\lambda) = \prod_{j=1}^{k}(\lambda - \gamma_j), \quad g_2(\lambda) = \prod_{j=1}^{p}(\lambda - \alpha_j), \quad g_3(\lambda) = \prod_{j=1}^{q}(\lambda - \beta_j).$$

Note that $\deg f = n - k$, $\deg g_1g_2g_3 = n - k - 1$. According to Lemma 6 there exists $a \in \mathbb{R}$, $x_1, x_2, x_3 \in \mathbb{R}^+$ and monic real polynomials h_1, h_2, h_3, with $\deg h_i = (\deg g_i) - 1$, having all its roots real, the roots of h_i strictly interlacing with the roots of g_i such that

$$f(\lambda) = (\lambda - a)g_1g_2g_3 - x_1h_1g_2g_3 - x_2h_2g_1g_3 - x_3h_3g_1g_2. \tag{5}$$

Let $y_1 = \frac{1}{2}x_1 + x_2$, $y_2 = \frac{1}{2}x_1 + x_3$ and

$$k_1 = \frac{1}{y_1}\left(\frac{x_1}{2}h_1g_2 + x_2h_2g_1\right), \quad k_2 = \frac{1}{y_2}\left(\frac{x_1}{2}h_1g_3 + x_3h_3g_1\right). \tag{6}$$

We may write (5) in the following way:

$$f(\lambda) = (\lambda - a)g_1g_2g_3 - y_1k_1g_3 - y_2k_2g_2. \tag{7}$$

Note that

$$\begin{aligned}\deg k_1 &= \max\{(\deg g_1) - 1 + \deg g_2, (\deg g_2) - 1 + \deg g_1\}\\ &= (\deg g_1 + \deg g_2) - 1,\end{aligned}$$

$$\begin{aligned}\deg k_2 &= \max\{(\deg g_1) - 1 + \deg g_3, (\deg g_3) - 1 + \deg g_1\}\\ &= (\deg g_1 + \deg g_3) - 1.\end{aligned}$$

By Lemma 1 the coefficients of the partial fraction decomposition of $h_1/g_1, h_2/g_2$ and h_3/g_3 are positive real numbers. But from (6) we have

$$\frac{k_1}{g_1g_2} = \frac{1}{y_1}\left(\frac{x_1}{2}\frac{h_1}{g_1} + \frac{h_2}{g_2}\right), \quad \frac{k_2}{g_1g_3} = \frac{1}{y_2}\left(\frac{x_1}{2}\frac{h_1}{g_1} + \frac{h_3}{g_3}\right)$$

and therefore the coefficients of the partial fraction decomposition of k_1/g_1g_2 and k_2/g_1g_3 are also positive real numbers. Using again Lemma 1, k_1 and k_2 have only real roots strictly interlacing with the roots of g_1g_2 and g_1g_3, respectively.

Now there exist real symmetric tridiagonal matrices A_1, A_2 of orders $k+p$ and $k+q$ and characteristic polynomials g_1g_2 and g_1g_3, respectively, and such that $A_1(p+k)$ (the submatrix obtained from A_1 by deleting the last row and column) has characteristic polynomial k_1 and $A_2(1)$ has characteristic polynomial k_2 (see e.g. [1, Section 3] and references therein).

Define a matrix $A = [a_{ij}]$ of order n in the following way:

$$A\big[1, \ldots, k+p\big] = A_1, \quad A\big[k+p+2, \ldots, n\big] = A_2$$

(note that $n - (k+p+1) = k+q$) $a_{p+k+1\ p+k+1} = a$ (as given in (5)),

$$a_{p+k-1\ p+k} = a_{p+k\ p+k-1} = \sqrt{y_1}, \quad a_{p+k+1\ p+k+2} = a_{p+k+2\ p+k+1} = \sqrt{y_2},$$

zero in all the remaining positions of A.

From the Laplace expansion of $\det(\lambda I - A)$ along the $p+q+1$ row it follows that the characteristic polynomial of A is

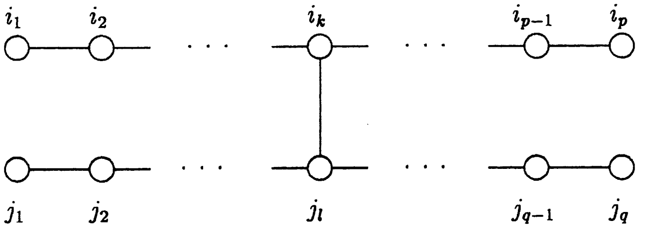
$$(\lambda - a)g_1^2g_2g_3 - y_1k_1g_1g_3 - y_2k_2g_1g_2.$$

But from (7) this polynomial is $g_1 f$, that is

$$\sigma(A) = \big\{\gamma_1, \ldots, \gamma_k, \lambda_{k+1}, \ldots, \lambda_n\big\}. \qquad \square$$

4. Double paths

We refer to a tree whose path cover number is 2 as a *double path*. In the spirit of the notion of a path tree from [5], a double path G appears as



in which the only constraint on the connecting edge $\{i_k, j_l\}$ is that not both $k \in \{1, p\}$ and $l \in \{1, q\}$. The upper (i) path has $k-1$ vertices to the left of the connecting vertex and another $p-k$ vertices to the right; set $s_1 = \min\{k-1,\ p-k\}$. Similarly, set $s_2 = \min\{l-1, q-l\}$.

If $A \in H(G)$, the maximum multiplicity of an eigenvalue of A is 2 by Theorem 2, but how many multiplicity 2 eigenvalues may $A \in H(G)$ have? Since G has $p+q$ vertices and the length of the longest path in G is $\max\{p, q, p+q-(s_1+s_2)\}$, $A \in H(G)$ has at least $\max\{p, q, p+q-(s_1+s_2)\}$ distinct eigenvalues and thus at most

$$s = \min\big\{q, p, s_1+s_2\big\}$$

multiplicity 2 eigenvalues. Using the strategy from Section 3, we may show that any number up to s multiplicity 2 eigenvalues is possible.

Theorem 8. *Let G be a double path whose paths have p and q vertices, respectively, and define s_1, s_2 and s as above. Then $L(G)$ consists of all partitions of $p+q$ into parts at most* 2 *with at most s* 2's.

Proof. First construct a real symmetric matrix A, with $G(A) = G$ and having t double eigenvalues, with $t \leqslant s$. Assume wlog that $p \leqslant q$.

Case 1. $s_1 + s_2 = s$. Let t_1, t_2 be integers such that $0 \leqslant t_1 \leqslant s_1$, $0 \leqslant t_2 \leqslant s_2$, $t_1 + t_2 = t$. Pick numbers $\lambda_1, \ldots, \lambda_{t_1}$, and $\lambda_{t_1+1}, \ldots, \lambda_{t_1+t_2}$ to be distinct and in such away that between any two λ's of the first set there is at most one λ of the second set and vice versa. Theorem 7 shows that it is possible to assign $\lambda_1, \ldots, \lambda_{t_1}$ as eigenvalues of $A[i_1, \ldots, i_{k-1}]$, $A[i_{k+1}, \ldots, i_p]$ and of $A[j_1, \ldots, j_q]$ and assign $\lambda_{t_1+1}, \ldots, \lambda_{t_1+t_2}$ as eigenvalues of $A[j_1, \ldots, j_{l-1}]$, $A[j_{l-1}, \ldots, j_q]$ and of $A[i_1, \ldots, i_p]$. Now if we delete i_k, $A[i_k]$ has eigenvalues $\lambda_1, \ldots, \lambda_t$ as eigenvalues with multiplicity 3 and so, by interlacing, A has these eigenvalues with multiplicity greater than or equal to 2. According to Theorem 2 this multiplicity is 2.

Case 2. $p = s$. Assign as before, except now that the second group of λ's has only $p - s_1$ elements.

Second, there can be no more double eigenvalues: Suppose that i_k is a Parter vertex for t_1 double eigenvalues of some real symmetric matrix A with $G(A) = G$ and j_l is a *Parter* vertex for other t_2 double eigenvalues of A. Then there are t_1 eigenvalues that occur in each of $A[i_1, \ldots, i_{k-1}]$, $A[i_{k+1}, \ldots, i_p]$ and $A[j_1, \ldots, j_q]$ and other (distinct) t_2 that occur in each of $A[j_1, \ldots, j_{l-1}]$, $A[j_{l-1}, \ldots, j_q]$ and of $A[i_1, \ldots, i_p]$. So $t_1 \leqslant s_1$, $t_2 \leqslant s_2$, $t_1 + t_2 \leqslant p$, $t_1 + t_2 \leqslant q$, completing the proof. $\square$

Remark. It follows from Theorem 8 that if G is a double path, any $\tilde{p} \in L(G)$ has at least $n - 2s$ 1's, in which n is the total number of vertices and s is as above. Of course, this number could be quite large if there is a big disparity in the lengths of the two paths of G or if the connecting edge is far off center of the two paths.

Again for any $\tilde{p} \in L(G)$ the multiple eigenvalues may take on any numerical values, subject to the distinctness (and order, which is irrelevant here) constraints.

5. Generalized stars

A *star* is a tree on n vertices with one vertex of degree $n-1$ and $n-1$ vertices of degree 1. Though it includes some subtleties, $L(T)$ may be determined for a star on n vertices. Our purpose here is to determine $L(T)$ whenever T is a *generalized star*: exactly one vertex of degree greater than 2. Such a tree conveniently has only one possible Parter vertex (the central one) and may be parametrized in terms of the number and lengths of paths ("arms") emanating from the central vertex. Let S be a generalized star on n vertices: then S is completely described by the number s of arms and the lengths (by the number of vertices not including the center) of those arms $l_1 \geqslant l_2 \cdots \geqslant l_s \geqslant 1$. Note that $1 + \sum_{i=1}^{s} l_i = n$. To describe $L(S)$, it is convenient to use the partition of $n-1$ conjugate to $l_1, \ldots, l_s$; thus, define m_t to be the number of l_i's that are at least t. Then $m_1 = s$ and $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{l_1} \geqslant 1$. Since the central vertex is the only one that may be the Parter vertex for any multiple eigenvalue, we may achieve a given list of multiplicities only by allocating the multiple eigenvalues among the arms. For each particular multiple eigenvalue (e.g. λ with multiplicity r), λ must occur once on $r+1$ of the arms. Thus, the multiplicities are constrained by the number of arms s, the total number of multiple eigenvalues and, of course, the specific lengths of the arms; but, this is all. Recall that a real vector u is majorized by another real vector v, both of whose components lie in descending order, if:

$$\begin{aligned} u_1 &\leqslant v_1 \\ u_1 + u_2 &\leqslant v_1 + v_2 \\ &\vdots \\ u_1 + u_2 + \cdots + u_c &= v_1 + v_2 + \cdots + v_d \end{aligned}$$

in which u has c components and v has d components. There are $\max\{c, d\}$ lines above and only the last one is required to be an equality. In this event, we write $u \preceq v$.

Theorem 9. *Let G be a generalized star on n vertices with arms $G_1, \ldots, G_s$, of lengths $l_1 \geqslant l_2 \cdots \geqslant l_s \geqslant 1$, respectively, $n = 1 + \sum_{i=1}^{s} l_i$. Then $(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_k) \in L(G)$ if and only if:*
(a) $k \geqslant l_1 + l_2 + 1$,
(b) $\sum \tilde{p}_i = n$,
(c) $\tilde{p}_h = \tilde{p}_{h+1} = \cdots = \tilde{p}_k = 1, \quad h = \left\lceil \frac{k+1}{2} \right\rceil$, *and*
(d) $(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{k-l_1-1}) \preceq \big(m_1 - 1, \ldots, m_{l_1} - 1\big)$.

Proof. We first note the necessity of the stated conditions: (b) simply says that there must be n eigenvalues, counting multiplicity; and (a) follows from Theorem 3, since the right-hand side of (a) is the diameter of G. By Theorem 1, any multiple eigenvalue with multiplicity $\tilde{p}_i > 1$ must be an eigenvalue of $\tilde{p}_i + 1$ of the principal submatrices associated with the arms of G (because each of these is irreducible tridiagonal and thus has no multiplicity greater than 1). Thus, there is a cost of 1 from the arm lengths for each multiple eigenvalue, and keeping in mind the central vertex as well, this means that there are more 1's than multiple eigenvalues among the $\tilde{p}$'s, which is (c). Similar counting argument shows that the number of 1's among the $\tilde{p}$'s is also greater than l_1. To prove (d) suppose that v is the central vertex of

G and denote by $\mu_1, \ldots, \mu_r$ the distinct eigenvalues of $A(v)$; put $\tilde{q}_i = m_{A(v)}(\mu_i)$, $i = 1, \ldots, r$ and suppose that the μ's are ordered in such a way that $\tilde{q}_1 \geqslant \cdots \geqslant \tilde{q}_r$. Now define a (0, 1)-matrix $T = [t_{ij}]$, of size $s \times r$, in the following way: $t_{ij} = 1$ if μ_j is an eigenvalue of $A[G_i]$, the submatrix of A corresponding to the arm G_i of G (whose length is l_i); $t_{ij} = 0$ otherwise. As the eigenvalues of a tridiagonal Hermitian matrix are all distinct, the sum of the elements of the i row of T is l_i (the length of G_i) while the sum of the elements of the j column is $\tilde{q}_j$. By the Gale–Ryser theorem (see e.g. [7, p. 176]) we have

$$(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_r) \preceq \left(m_1, m_2, \ldots, m_{l_1}\right). \tag{8}$$

Let k' be the number of $\tilde{p}$'s that are greater than or equal to 2. From Theorem 1 we have $\tilde{q}_i = \tilde{p}_i + 1$, $i = 1, \ldots, k'$, and so from (8)

$$\sum_{i=1}^{j} \tilde{p}_i \leqslant \sum_{i=1}^{j} (m_i - 1) \quad \text{for } j = 1, \ldots, \min\{k', l_1\}. \tag{9}$$

As noted above the number of $\tilde{p}$'s equal to 1 is greater than l_1. This means that $k' \leqslant k - l_1 - 1$ and

$$\sum_{i=1}^{k-l_1-1} \tilde{p}_i = \sum_{i=1}^{l_1} (m_i - 1). \tag{10}$$

From this equality follows

$$\sum_{i=1}^{j} \tilde{p}_i \leqslant \sum_{i=1}^{l_1} (m_i - 1) \quad \text{for } j = 1, \ldots, k - l_1 - 1 \tag{11}$$

and, if $k - l_1 - 1 < l_1$,

$$m_i - 1 = 0 \quad \text{for } i = k - l_1, \ldots, l_1. \tag{12}$$

If $k' \geqslant l_1$, then (9)–(12) prove (d). Suppose now that $k' < l_1$. It remains to prove that

$$\sum_{i=1}^{j} \tilde{p}_i \leqslant \sum_{i=1}^{j} (m_i - 1) \quad \text{for } j = k' + 1, \ldots, \ \min\{k - l_1 - 1, l_1\}. \tag{13}$$

Let j be an integer such that $k' < j \leqslant k - l_1 - 1$. If $m_j - 1 > 0$, then for $i = k' + 1, \ldots, j$ we have $\tilde{p}_i = 1 \leqslant m_i - 1$ and so (13) follows from (9). If $m_j - 1 = 0$ also $m_i - 1 = 0$ for $i = j + 1, \ldots, l_1$ and so $\sum_{i=1}^{j}(m_i - 1) = \sum_{i=1}^{l_1}(m_i - 1)$. Eq. (13) follows now from (11). This completes the proof of (d).

For sufficiency, let k' be, as above, the number of $\tilde{p}$'s that are greater than or equal to 2. For $i = 1, \ldots, k'$ define $\tilde{q}_i = \tilde{p}_i + 1$. (c) implies that $\sum_{i=1}^{k'} \tilde{q}_i \leqslant n - 1$. Let $r = k' + n - 1 - \sum_{i=1}^{k'} \tilde{q}_i$ and, if $r > k'$, put $\tilde{q}_i = 1$ for $i = k' + 1, \ldots, r$. Now we have $\sum_{i=1}^{r} \tilde{q}_i = n - 1$.

We are going to prove that the sequence of $\tilde{q}$'s is majorized by the sequence of m's. From (d) we have

$$\sum_{i=1}^{k-l_1-1} \tilde{p}_i = \sum_{i=1}^{l_1}(m_i - 1).$$

The right-hand side of this equality is $n - l_1 - 1$ and so it follows from (b) that $\tilde{p}_{k-l_1} = \cdots = \tilde{p}_k = 1$. This means that $k' \leqslant k - l_1 - 1$ and we have from (d)

$$\sum_{i=1}^{j} \tilde{q}_i \leqslant \sum_{i=1}^{j} m_i \quad \text{for } j = 1, \ldots, \min\{k', l_1\}. \tag{14}$$

The sum of all m's is $n - 1$ and so

$$\sum_{i=1}^{r} \tilde{q}_i = \sum_{i=1}^{l_1} m_i. \tag{15}$$

From this equality follows

$$\sum_{i=1}^{j} \tilde{q}_i \leqslant \sum_{i=1}^{l_1} m_i \quad \text{for } j = 1, \ldots, r. \tag{16}$$

If $k' < l_1$, then the fact that, for $i > k'$, $\tilde{q}_i = 1$ together with (15) gives

$$\sum_{i=1}^{j} \tilde{q}_i \leqslant \sum_{i=1}^{j} m_i \quad \text{for } j = 1, \ldots, \min\{r, l_1\} \tag{17}$$

(in fact (15) and (17) imply that $r \geqslant l_1$).

Now (14)–(17) are precisely

$$(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_r) \preceq \left(m_1, m_2, \ldots, m_{l_1}\right).$$

Again by the Gale–Ryser theorem [7, p. 176] there exists a (0, 1)-matrix $T = [t_{ij}]$, of size $s \times r$ such that the sum of the elements in the row i is l_i, while the sum of the elements of the j column is $\tilde{q}_j$.

Pick r distinct real numbers $\mu_1, \ldots, \mu_r$ and for each i, $1 \leqslant i \leqslant s$, construct an l_i-by-l_i irreducible tridiagonal matrix A_i such that μ_j is an eigenvalue of A_i if and only if $t_{ij} = 1$. This construction is possible because for each i there are exactly l_i (distinct) μ's for which $t_{ij} = 1$ and because an l_i-by-l_i irreducible tridiagonal matrix may be constructed with any l_i distinct eigenvalues. Each μ_j will be an eigenvalue of exactly $\tilde{q}_j$ of those matrices.

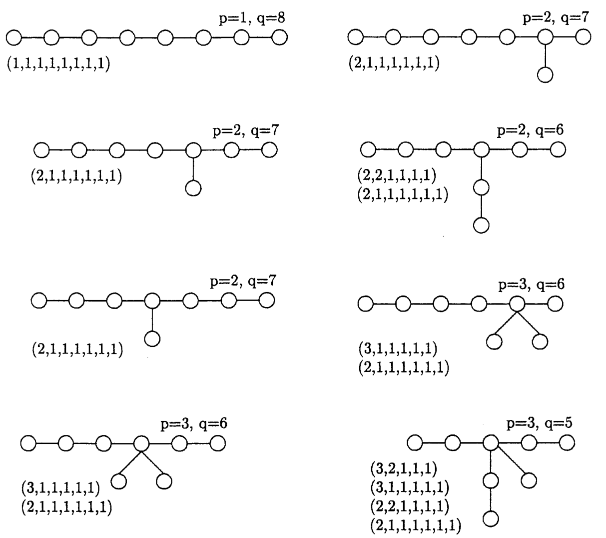
Let A be any matrix with graph G and such that $A[G_i]$ is, up to a permutation similarity, A_i. Now, in any Hermitian matrix A with graph G and principal submatrices as constructed associated with the arms, the multiplicity of μ_i, $i = 1, \ldots, k'$, is at least $\tilde{p}_i$ by the interlacing inequalities [4, Chapter 4] and at most $\tilde{p}_i$ by Theorem 1. Thus A realizes all $\tilde{p}_i > 1$ and all $\tilde{p}_i = 1$ (all eigenvalues of A besides the μ_i's associated with the $\tilde{p}_i > 1$ must have multiplicity 1 by Theorem 1 because no other eigenvalues appear on more than two submatrices and the central vertex is the only Parter vertex). Thus, the stated conditions are also sufficient, completing the proof. □

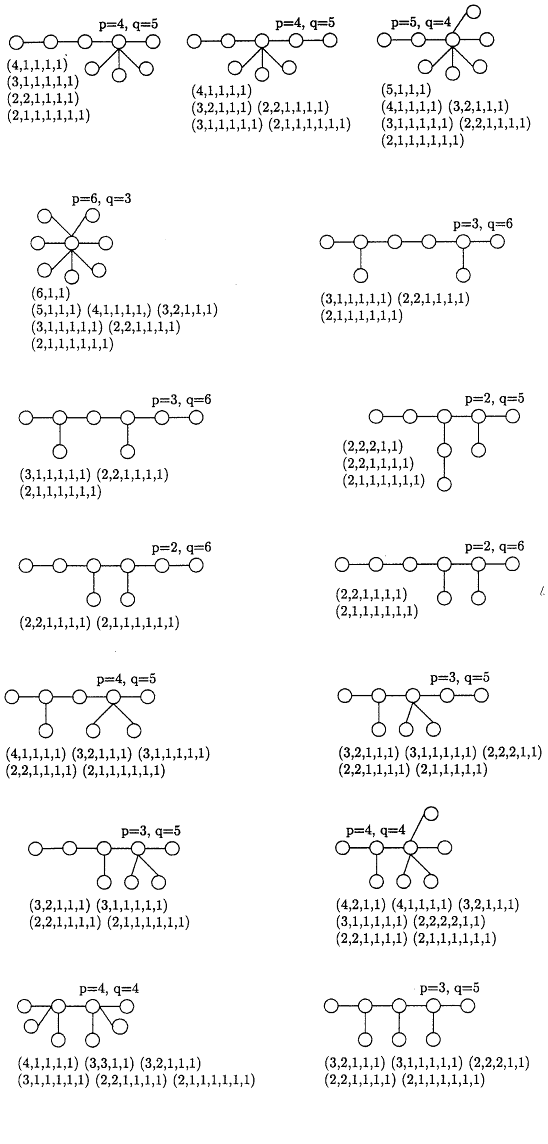
Remarks. Relative to condition (c) of Theorem 9, it may happen that more of the $\tilde{p}_i$'s are 1 and more may be forced to be 1 by condition (d). For example the generalized star on 13 vertices with arm lengths 3–5 requires at least seven 1's (not $\lceil \frac{5+4+1+1}{2} \rceil = 6$), as no multiplicity can exceed $2 = m_1 - 1$ and the four 2's are not allowed by the majorization. In general a double star with arm lengths $l_1 \geqslant l_2 \cdots \geqslant l_s$ requires $1 + l_1 + \max\{0, l_2 - \sum_{i=3}^{s} l_i\}$ 1's.

We also note that (a) of Theorem 9 is implied by (b), (c) and (d) (which are independent), but we have included it for clarity. It easily seen, using (d), that there are at least $l_1 + 1$ ones in $\tilde{p} \in L(G)$ (not comparable to (c)) and this statement could be substitued for (b).

6. $L(G)$ for 8-vertex trees

The construction tools given in Section 3, together with the limitations on multiplicities we have discussed, allow the determination of $L(G)$ for many trees besides the double paths and generalized stars we have discussed. Of course a large fraction of the trees on modest numbers of vertices are in one of those classes. We have used these ideas to determine $L(G)$ for each of the 23 eight-vertex trees and we report the results in this section. For each tree, we list each element of $L(G)$ as a tuple, generally omitting (1, 1, 1, 1, 1, 1, 1, 1), which is always present. In each case, the multiplicity list presented can be constructed and other lists have been ruled out using ideas discussed herein. We note that such lists for trees on fewer than eight vertices have been given in [6] and that many trees on nine (and more) vertices could be done as well.





The values p and q attached to each tree are, respectively, the path cover number and the diameter (see Theorem 3). We wish to emphasize that through 8-vertex trees, any list of multiplicities allowed by the constraints given in Section 2 actually occur, and we conjecture that this remains true in general. It is not, however, the case that any multiplicity allowed by the values of p and q (through Theorems 2 and 3) occurs. There are pairs of trees in our list with common values for p and q but different lists of possibilities.

References

[1] M.T. Chu, Inverse eigenvalues problems, SIAM Rev. 40 (1998) 1–39.
[2] J. Genin, J.S. Maybee, Mechanical vibration trees, J. Math. Anal. Appl. 45 (1974) 746–763.
[3] W. Ferguson, The construction of Jacobi and periodic Jacobi matrices with prescribed spectra, Math. Comp. 35 (1980) 1203–1220.
[4] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[5] C.R. Johnson, A. Leal Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, Linear and Multilinear Algebra 46 (1999) 139–144.
[6] A. Leal-Duarte, C.R. Johnson, On the minimum number of distinct eigenvalues for a symmetric matrix whose graph is a given tree, Math. Inequal. Appl. (to appear).
[7] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.
[8] P. Nylen, Minimum-rank matrices with prescribed graph, Linear Algebra Appl. 248 (1996) 303–316.
[9] S. Parter, On the eigenvalues and eigenvectors of a class of matrices, J. Soc. Indust. Appl. Math 8 (1960) 376–388.
[10] G. Wiener, Spectral multiplicity and splitting results for a class of qualitative matrices, Linear Algebra Appl. 61 (1984) 15–29.